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EFFECT OF VISCOSITY ON DYNAMICS OF BUBBLE PERTURBATIONS IN A LIQUID

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The dynamics of a bubble in a low-viscosity liquid is considered. The equation of dynamics of small perturbations with allowance for viscosity effects is determined.

Key words: *bubble, viscosity, perturbation, equation of dynamics.*

Let us consider a bubble with a variable radius in a low-viscosity liquid. Let the shape of the bubble surface S be close to spherical. The goal of the present paper is to derive the equation of dynamics of surface perturbations.

Let us introduce a spherical coordinate system (r,θ,φ) with the origin in the sphere center. The expression for the bubble-surface radius r_0 includes a small perturbation proportional to the spherical harmonic Y_n:

$$r_0(\theta,\varphi,t) = R(t) + r_1(t) + a(t)Y_n(\theta,\varphi), \qquad a \ll R. \tag{1}$$

Here, R is the radius of a sphere with an equivalent volume and a is the perturbation amplitude; the integral of Y_n^2 over a unit sphere is assumed to be equal to unity. In what follows, the argument t at the functions R, r_1, and a is omitted. It follows from the expression for the bubble volume $V = (4/3)\pi R^3$ that the value of the variable is $r_1 = -a^2/(4\pi R)$.

The unperturbed velocity field is potential. Let the perturbation of the velocity field around the bubble be also close to potential. This is possible if the product of the kinematic viscosity ν and the characteristic time τ is small: $\nu\tau \ll l^2$ ($l = \pi R/n$ is the characteristic distance).

The velocity potential $\boldsymbol{v} = \nabla\Phi$ is found by solving the Neumann problem

$$\Delta\Phi = 0, \qquad \frac{\partial\Phi}{\partial n} = \frac{\partial r_0}{\partial t}\, n_r, \qquad \Phi \to 0, \quad r \to \infty$$

in an approximate form:

$$\Phi = -\dot{R}\,\frac{R^2}{r} - \frac{1}{n+1}\Big(\dot{a} + 2\dot{R}\,\frac{a}{R}\Big)\Big(\frac{R}{r}\Big)^{n+1} RY_n. \tag{2}$$

Here, the coefficient at $1/r$ is valid with accuracy to small $\dot{R}a^3/R^2$ and $\dot{a}a^2/R$.

To describe the perturbation dynamics, we use an energetic approach similar to that used in [1] to describe the dynamics of a system of bubbles in a low-viscosity liquid. The generalized forces in the Lagrange equations can be expressed in terms of the rate of energy dissipation E in the potential velocity field:

$$\frac{d}{dt}\,\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \frac{1}{2}\,\frac{\partial E}{\partial \dot{q}_i}, \qquad i = 1,2. \tag{3}$$

The generalized coordinates are $q_1 = a$ and $q_2 = R$. The Lagrange function is $L = T_f - \sigma S$ (T_f is the kinetic energy of the liquid and σ is the surface tension). The kinetic energy and the dissipation rate are determined by the formulas

$$T_f = -\rho \int_S \Phi\,\frac{\partial\Phi}{\partial n}\,dS, \qquad E = 2\mu \int_\Omega \varepsilon_{ij}\varepsilon_{ij}\,d\Omega, \qquad \varepsilon_{ij} = \frac{1}{2}\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big), \tag{4}$$

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where ρ is the liquid density, μ is the dynamic viscosity, and Ω is the domain occupied by the liquid; summation is performed over repeated subscripts. The integral in the formula for E is reduced to the surface integral

$$E = -\mu \int\limits_S \frac{\partial v^2}{\partial n}\, dS. \tag{5}$$

In calculating the integrals, we use approximate expressions for the components of the normal to the bubble surface:

$$n_r = 1 - \frac{1}{2}\,(n_\theta^2 + n_\varphi^2), \qquad n_\theta = -\frac{a}{R}\,Y'_\theta, \qquad n_\varphi = -\frac{a}{R\sin\theta}\,Y'_\varphi. \tag{6}$$

Here, $Y = Y_n$. The area of the surface element is

$$dS = r_0^2(1 + (n_\theta^2 + n_\varphi^2)/2)\sin\theta\, d\varphi\, d\theta.$$

In what follows, we also use the integrals over a unit sphere S':

$$\int\limits_{S'} Y^2\, dS' = 1, \qquad \int\limits_{S'} \Big(Y_\theta'^2 + \frac{1}{\sin^2\theta}\, Y_\varphi'^2\Big)\, dS' = n(n+1).$$

The second integral is needed for calculating the energy dissipation rate E.

From Eqs. (1), (2), and (4), we find the kinetic energy, taking into account the principal terms with respect to the small amplitude a and velocity $\dot{a}$:

$$T_f = 2\pi\rho R^3\dot{R}^2 + \rho\,\frac{R^3}{2}\,\frac{\dot{a}^2}{n+1} - \rho\,\frac{n-1}{n+1}\,(a^2\dot{R}^2R + a\dot{a}\dot{R}R^2) + \ldots. \tag{7}$$

The potential energy (free surface energy) is determined as

$$\sigma S \approx 4\pi R^2\sigma + (n-1)(n-2)\sigma a^2/2. \tag{8}$$

Calculating E requires more transformations than calculating the kinetic energy T; therefore, we give only intermediate expressions:

$$\begin{gathered}
v^2 = \dot{R}^2\,\frac{R^4}{r^4} + 2\dot{R}u\Big(\frac{R}{r}\Big)^{n+4} Y + u^2\Big(\frac{R}{r}\Big)^{2n+4}\Big(Y^2 + \frac{1}{(n+1)^2}\Big(Y_\theta'^2 + \frac{1}{\sin^2\theta}\,Y_\varphi'^2\Big)\Big);\\
-\frac{\partial v^2}{\partial r} = 4\,\frac{\dot{R}^2}{R}\Big(1 - 5\varepsilon Y + 15\varepsilon^2Y^2 - 5\,\frac{r_1}{R}\Big) + \frac{2\dot{R}u}{R}\,(n+4)(Y - \varepsilon Y^2(n+5))\\
+ \frac{2u^2}{R}\,(n+2)\Big(Y^2 + \frac{1}{(n+1)^2}\Big(Y_\theta'^2 + \frac{1}{\sin^2\theta}\,Y_\varphi'^2\Big)\Big) \qquad \text{for} \quad r = r_0 \quad \Big(\varepsilon = \frac{a}{R}\Big),\\
-\frac{1}{R}\,\frac{\partial v^2}{\partial\theta} = -\frac{2\dot{R}u}{R}\,Y'_\theta, \qquad -\frac{1}{R\sin\theta}\,\frac{\partial v^2}{\partial\varphi} = -\frac{2\dot{R}u}{R\sin\theta}\,Y'_\varphi \qquad \text{for} \quad r = r_0.
\end{gathered} \tag{9}$$

Here $u = \dot{a} + 2a\dot{R}/R$. We multiply each formula in (9) by the corresponding component of the normal (6) and sum up them to obtain $-\partial v^2/\partial n$. Calculating integral (5), we obtain an approximate expression for the energy dissipation rate:

$$E = 16\pi\mu\dot{R}^2R + 2\mu[(n+2)(2n+1)\dot{a}^2R + 2(n+2)(n-1)a\dot{a}\dot{R} - 2(n-1)(2n+1)a^2\dot{R}^2R^{-1}]/(n+1). \tag{10}$$

With allowance for Eqs. (7), (8), and (10), the Lagrange equations (3) yield the following equation for the harmonic perturbation amplitude $a = a_n$:

$$R\ddot{a}_n + 3\dot{R}\dot{a}_n + [(n^2-1)(n+2)\sigma/(\rho R^2) - (n-1)\ddot{R}]a_n = -2\nu(n+2)[(2n+1)R\dot{a}_n + (n-1)\dot{R}a_n]/R^2. \tag{11}$$

In the case with zero viscosity ($\nu = 0$), the right side of the dynamic equation (11), which is the contribution of viscosity to the bubble perturbation dynamics, is also equal to zero, and Eq. (11) coincides with the equation derived in [2].

Equation (11) is a corrected Eq. (2) from [3], where the additional term is redundant.

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